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The quantization of Gel'fand–Yaglom equations for free fields of arbitrary spin

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Abstract. We consider the Gel'fand–Yaglom equations for free fields of arbitrary spin, in the particular case when the field transforms according to a direct sum of inequivalent, irreducible finite representations of the proper Lorentz group. Under the assumption that the theory carries neither physical states of zero charge or energy density and that the mass–spin states are non-degenerate, we obtain the precise forms of the minimal and characteristic polynomials of the s blocks of the L_0 matrix, which are then used to obtain new necessary and sufficient conditions that the theory be quantizable. The representation according to which the field transforms can be depicted graphically in a simple way and we take advantage of this to use some simple ideas of graph theory to obtain our results. This graphical approach is useful in practical and theoretical considerations in the theory. One conclusion is that it will probably be necessary to allow repeated irreducible representations of the proper Lorentz group for theories of spin greater than eight to be quantizable.

1. Introduction

We consider relativistic free field theories based on the equation

$$(L_\mu \partial^\mu + i\chi)\psi = 0 \tag{1.1}$$

where ψ is the field variable, L_μ are matrices and χ is a real number and $\partial^\mu = \partial/\partial x_\mu$. We adopt the notations and conventions of Gel'fand and Yaglom (Gel'fand *et al* 1963), who have made a thorough study of the field theory of (1.1). Very little has been done on finding theories described by (1.1) which are quantizable. Recently Capri and Shamaly (1971) and Amar and Dozzio (1972a) have shown how special types of quantizable theories can be obtained, basically by requiring that the theory describes a unique mass state and that the corresponding nonzero eigenvalue of L_0 should be non-degenerate except for the natural degeneracy due to spin. Amar and Dozzio have in a second paper (Amar and Dozzio 1972b) investigated theories in which L_0 is allowed to be non-diagonalizable, for as Gel'fand and Yaglom have shown it is only in this case that quantizable theories are possible for spins greater than one. The spin block components of L_0 (the so called 's blocks') are studied individually, and the minimum spin for which a spin block is diagonalizable governs the structure of the possible theories. If L_0 is non-diagonalizable then it is possible to get multiple mass states in the theory.

A feature of the theories of Capri *et al* is that ψ may transform according to a reducible representation $\mathscr{R}$ of the Lorentz group $\mathscr{L}$, which in general can contain repeated representations of the proper Lorentz group $\mathscr{L}_p$. When one does allow repeated representations, it is difficult to know where to stop, and it is interesting to know what theories are

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possible when we explicitly exclude such repeated representations—that is we assume that all the representations in $\mathscr{R}$ are inequivalent. In this paper we study such theories and their quantization in detail. We impose two further very reasonable requirements on our theories, namely that there are no physical states corresponding to zero charge or energy density, and that there are no two states with the same mass and spin (this is the non-degeneracy of mass–spin states of Amar and Dozzio, mentioned above). We study the properties of the s blocks for such theories in general, obtaining the exact form of their minimal and characteristic polynomials and use these to simplify the conditions of quantization, which we express directly in terms of the s blocks. A graphical approach to the Gel'fand–Yaglom equation is presented which takes advantage of the simple graphical form of the representation $\mathscr{R}$ carried by solutions ψ of the equations. This approach is useful in the development of the general theory and in studying particular cases.

2. The Gel'fand–Yaglom theory

We assume that the field variable ψ transforms according to some finite dimensional representation $\mathscr{R}$ of the homogeneous Lorentz group $\mathscr{L}$. In general $\mathscr{R}$ is taken to be a direct sum of irreducible finite dimensional representations τ_i of $\mathscr{L}_p$, the proper Lorentz group, and we assume that all of the τ_i are inequivalent. Then any representation τ_i in $\mathscr{R}$ is uniquely specified by an ordered pair of numbers $(l_0^{(i)}, l_1^{(i)})$ which are either both integers or both half-odd integers, and $l_1^{(i)} > |l_0^{(i)}|$. The representation conjugate to τ_i, denoted τ_i^* is then specified by $(-l_0^{(i)}, l_1^{(i)})$.

A canonical basis $\{\xi_{sm}^{\tau_i}\}$ where $\tau_i \in \mathscr{R}$; $m = s, s-1, \ldots, -(s-1), -s$;

$$s = |l_0^{(i)}|, |l_0^{(i)}|+1, \ldots, l_1 - 1,$$

can be constructed for $\mathscr{R}$ as described in (1). Gel'fand and Yaglom have determined the most general form for the L_μ in the canonical basis, such that (1.1) is covariant under $\mathscr{L}$ and is derivable from a real invariant lagrangian density. It turns out that only L_0 is important, as the $L_i, i = 1, 2, 3$ can all be expressed in terms of L_0. In the canonical basis L_0 has the form

$$L_0 = [C_{sms'm'}^{\tau\tau'}] \tag{2.1}$$

where, due to covariance under $\mathscr{L}_p$,

$$C_{sms'm'}^{\tau\tau'} = C_s^{\tau\tau'} \delta_{ss'} \delta_{mm'} \tag{2.2}$$

and the $C_s^{\tau\tau'}$ are only nonzero if the representations τ and τ' are 'interlocked' or 'linked', ie τ and τ' are such that $l_0' = l_0 \pm 1$, or $l_1' = l_1 \pm 1$. The nonzero $C_s^{\tau\tau'}$ have the form

$$\frac{(l_0', l_1') = (l_0 + 1, l_1)}{C_s^{\tau\tau'} = \rho(s, l_0) C^{\tau\tau'}}$$

$$C_s^{\tau'\tau} = \rho(s, l_0) C^{\tau'\tau} \tag{2.3}$$

$$\frac{(l_0', l_1') = (l_0, l_1 + 1)}{C_s^{\tau\tau'} = \rho(s, l_1) C^{\tau\tau'}}$$

$$C_s^{\tau'\tau} = \rho(s, l_1) C^{\tau'\tau} \tag{2.4}$$

where the $C^{\tau\tau'}$, $C^{\tau'\tau}$ are arbitrary complex numbers, independent of s and m, and where

$\rho(s,n) = \sqrt{(s+n-1)(s-n)}$. The further conditions for space reflection covariance and lagrangian origin show that all the elements $C_s^{\tau\tau'}$, $C_s^{\tau'\tau}$, $C_s^{\tau\tau''}$, $C_s^{\tau''\tau'}$ depend on the single complex number $C^{\tau\tau'}$, and they are all nonzero if and only if $C^{\tau\tau'}$ is.

If (1.1) is to be derivable from a real invariant lagrangian density, then we need an invariant non-degenerate hermitian form in the field variables, with which to construct this. As Gel'fand and Yaglom have shown, the most general such form in $\mathscr{R}$ is given in the canonical basis by

$$\begin{aligned}(\psi_1, \psi_2) &= \psi_2^\dagger \Lambda \psi_1 \\ &= \sum_{\tau s m} a^{\tau\tau'} e_s x_{sm}^\tau \overline{y_{sm}^{\tau'}}\end{aligned} \tag{2.5}$$

where $\psi_1 = (x_{sm}^\tau)$, $\psi_2 = (y_{sm}^\tau)$ and $e_s = (-1)^{[s]}$, and Λ is non-singular. The $a^{\tau\tau'}$ satisfy $a^{\tau\tau'} = a^{\tau'\tau}$ and in fact we can always choose a canonical basis such that $a^{\tau\tau'} = \pm 1$. (1.1) can now be derived from the lagrangian density:

$$L[\psi(x)] = \frac{1}{\mathrm{i}} \psi^\dagger \Lambda (L_\mu \partial^\mu + \mathrm{i}\chi)\psi \tag{2.6}$$

which is invariant. Reality of $L[\psi(x)]$ leads to, in particular

$$L_0^\dagger \Lambda = \Lambda L_0. \tag{2.7}$$

Now (2.2) shows that L_0 can be put in block diagonal form, by grouping together all basis vectors ξ_{sm}^τ, $m = -s, -(s-1), \ldots, s-1, s$ corresponding to the same value of s. These blocks are called 's blocks' and have the typical form

$$\Lambda_s = [C_s^{\tau\tau'}].$$

We always suppress the m dependence in the s blocks. Each of the elements in an s block is in fact a $(2s+1)\times(2s+1)$ scalar matrix and so can be treated like a scalar.

A nonzero eigenvalue μ of an s block is a $2s+1$ fold eigenvalue of L_0 and specifies a state of rest mass χ/μ and spin s. In this way, the s blocks of L_0 provide a list of all the states described by the field theory. Further, we shall see later that the conditions for quantization of the theory can be expressed in terms of relations on the s blocks. For the remainder of this section we restrict ourselves to integral spin theories, for the purposes of illustration, the modifications for half-odd integral spin theories being fairly obvious.

All finite dimensional representations of $\mathscr{L}_p$ which carry an integral spin representation can be obtained by letting $l_0 = 0, 1, 2, \ldots$ and $l_1 = 1, 2, \ldots$ where $l_1 > |l_0|$. Plotting these representations in the (l_0, l_1) plane gives a fan of points, and $\mathscr{R}$ can be completely specified by a finite subset of these points (figure 1). We denote a point $(l_0^{(i)}, l_1^{(i)})$ by the representation τ_i to which it corresponds. The s blocks of L_0 will only have rows and columns which correspond to representations τ_i which carry a spin s. If j is the maximum value of s occurring in the theory (ie the maximum possible spin), then inspection of the graph shows that these representations will be only those which lie in or on the rectangle:

$$\begin{aligned} l_0 &= -s, & l_0 &= s \\ l_1 &= s+1, & l_1 &= j+1. \end{aligned}$$

Consider this subset of points and construct an abstract linear graph as follows. Draw a directed branch from the point τ_i to τ_j if and only if $C_s^{\tau_i\tau_j}$, in the s block is nonzero. Since this only occurs if τ_i and τ_j are linked, the resulting graph will be a subgraph of a lattice type graph typified by figure 2. Associating the element $C_s^{\tau_i\tau_j}$ with the branch from

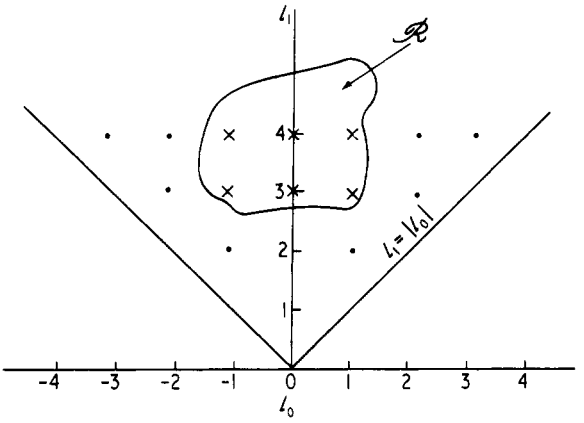

Figure 1.

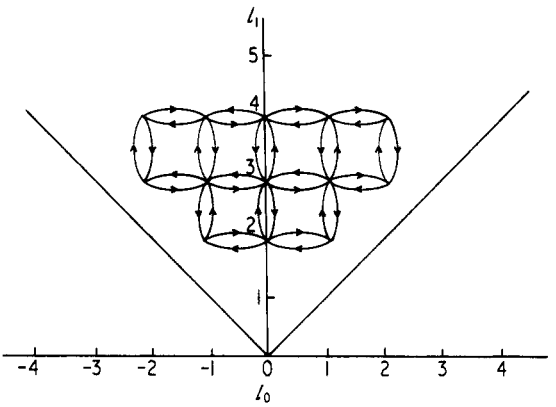

Figure 2.

τ_i to τ_j gives a pictorial representation of the s block which can be very useful in its construction. All the conditions on the $C_s^{\tau_i\tau_j}$ due to covariance and lagrangian origin can be transposed onto the graph and then the s blocks for any representation $\mathscr{R}$ can be written down immediately from the graphical plot of $\mathscr{R}$. Further, the graph corresponding to a given s block can be used as a visual aid in studying the properties of the s blocks, and dealing with particular theories. Because of the simple form of the graph, this is often useful in the following sections, where we first obtain the exact form of the characteristic and minimal polynomials of the s block, subject to certain assumptions, and then use these to obtain conditions on the s blocks for quantizable theories. We use two simple ideas from graph theory in the following:

(i) The graphical interpretation of the determinant of a matrix (Harary 1959).

(ii) The graphical interpretation of the elements of powers of a matrix.

For our purposes, the intuitive idea of a (directed) 'loop' in a graph will suffice. A 'path' in a directed graph is a connected sequence of branches all of like direction, while the 'length' of the path is the number of branches it contains. A loop is a closed path.

3. Characteristic and minimal polynomials of the s blocks

The results of this section apply to both integral and half-odd integral spin theories.

The determinant of a matrix can be interpreted graphically as follows. Let ζ be any directed graph with n nodes and $\mathcal{M}(\zeta)$ any $n \times n$ matrix which can only have nonzero elements in the positions i, j if there is a directed branch from node i to node j in ζ. Then each branch $i \to j$ of ζ represents a possible nonzero element m_{ij} of $\mathcal{M}(\zeta)$. Any term in the determinant $|\mathcal{M}(\zeta)|$ must be of the form

$$\pm m_{1i_1} m_{2i_2} \ldots m_{ni_n}$$

where $i_1 i_2 \ldots i_n$ is some permutation of the numbers 1 to n. Since any permutation can be written as a product of disjoint cycles, the nonzero terms of $|\mathcal{M}(\zeta)|$ must correspond to sets of disjoint cycles, or loops, of the graph ζ. Now consider a lattice type graph G, such as that of an s block, and with associated matrix $M(G)$. Let m_{ij} be the i, j element of $M(G)$, and consider the matrix $M(G) - tI$. This will be represented by a lattice graph of the same type as G, except that each node has a self loop, which corresponds to the nonzero diagonal element $-t$. Apart from these self loops, all loops of such a graph clearly have an even number of branches. The self loops have one branch. Any nonzero term of the expansion of $|M(G) - tI|$ must be of the form

$$\pm m_{i_1 j_1} m_{i_2 j_2} \ldots m_{i_p j_p} (-t)^r$$

where p must be even, since the m part of this term must correspond to a union of disjoint loops with an even number of branches. The $(-t)^r$ part represents the self loops of r nodes. Since $r + p = n$, r is even or odd with n, and it follows that for such a lattice type graph $|M(G) - tI|$ has the form $tP_1(t^2)$ if n is odd and $P_2(t^2)$ if n is even, where P_1 and P_2 are polynomials. The above derivation also suggests a method of writing down the characteristic polynomial of $M(G)$ directly from the graph (Cox 1972).

It follows from the above that we can write the characteristic polynomial $\Delta(t)$ of an $n \times n$ s block A in the form

$$\Delta(t) = t^p \prod_{i=1}^{k} (t^2 - m_i^2)^{r_i} \tag{3.1}$$

where $p + \sum_{i=1}^{k} r_i = n$, the m_i are all distinct, and none of the r_i are zero. Now the minimal polynomial of A must contain the same irreducible factors as the characteristic polynomial and so will have a form similar to that above. However, as Udgaonkar (1952) has shown, a necessary and sufficient condition for a theory to have no physical states (those corresponding to nonzero eigenvalues of L_0) with a zero charge or energy density is that the minimal polynomial of L_0 should have no repeated factors corresponding to nonzero eigenvalues. Since the minimal polynomial of L_0 is the least common multiple of the minimal polynomial of the s blocks, it follows that for such 'physical' theories the minimal polynomials of the s blocks must contain no repeated factors corresponding to nonzero eigenvalues and must therefore be of the form

$$m(t) = t^q \prod_{i=1}^{k} (t^2 - m_i^2) \tag{3.2}$$

where $q + 2k \leqslant n$.

We will now make a further reasonable assumption, namely that the only degeneracy possessed by a state characterized by a nonzero eigenvalue λ of an s block, shall be attributable to the spin components $m = s, s-1, \ldots, -(s-1), -s$. So, to each nonzero

eigenvalue of an s block there is exactly one independent eigenvector. With this assumption and with the minimal polynomial (3.2) it follows that $r_i = 1$ for all i, in (3.1). (To see this, consider for example the Jordan normal form of the s block, given (3.2) as the minimal polynomial and unity as the geometric multiplicity of each nonzero eigenvalue.) Thus the characteristic polynomial of an s block must be of the form

$$\Delta(t) = t^{n-2k} \prod_{i=1}^{k} (t^2 - m_i^2) \tag{3.3}$$

if the physical states are non-degenerate and the charge and energy densities are non-vanishing. The m_i in (3.3) are all distinct. Conversely, if the s block has minimal polynomial (3.2) and characteristic polynomial (3.3) then the physical states are non-degenerate and the charge and energy densities are nonzero.

Finally, for such theories, the eigenvalues $\pm m_i$ are all real, since from (2.7) and the fact that Λ does not mix up s subspaces (see (2.5)), we have

$$A_s^\dagger \Lambda_s = \Lambda_s A_s \tag{3.4}$$

for any s block A_s, where Λ_s is the restriction of Λ to the s subspace. Thus if m_i is a nonzero eigenvalue of A_s and ψ_i its eigenvector then from (3.4) we deduce

$$(m_i - \bar{m}_i)\psi_i^\dagger \Lambda \psi_i = 0.$$

For theories we are considering, with nonzero energy and charge, $\psi_i^\dagger \Lambda \psi_i \neq 0$ and so $m_i = \bar{m}_i$, all the eigenvalues m_i are real.

4. Quantization

The field energy and charge densities are the real quantities

$$H = \mathrm{i}(L_0\psi, \partial_0\psi) \tag{4.1}$$

$$\rho = (L_0\psi, \psi) \tag{4.2}$$

respectively. ψ can be expanded in eigenfunctions of ∂_μ as follows:

$$\psi = \sum_{kr} \{\phi_+(k, r) \exp[\mathrm{i}(k \cdot x + \omega_{kr}t)] + \phi_-(k, r) \exp[\mathrm{i}(k \cdot x - \omega_{kr}t)]\} \tag{4.3}$$

where $\phi_+(k, r)$ and $\phi_-(k, r)$ satisfy

$$(L \cdot k - \omega_{kr}L_0 + \chi)\phi_+(k, r) = 0 \tag{4.4}$$

$$(L \cdot k + \omega_{kr}L_0 + \chi)\phi_-(k, r) = 0 \tag{4.5}$$

and $\omega_{kr} = k_0 = [k^2 + (\chi/m_r)^2]^{1/2}$, m_r being the nonzero eigenvalues of L_0.

$k = (k_1, k_2, k_3)$ and $L = (L_1, L_2, L_3)$.

From (4.4) and (4.5) we obtain the orthogonality relations:

$$\phi_+^\dagger(k, r')\Lambda L_0 \phi_-(k, r) = 0 \qquad \text{for any } r, r' \tag{4.6}$$

and

$$\phi_+^\dagger(k, r')\Lambda L_0 \phi_+(k, r) = \phi_-^\dagger(k, r')\Lambda L_0 \phi_-(k, r) = 0 \tag{4.7}$$

if $\omega_{kr'} \neq \omega_{kr}$.

Substituting (4.3) into (4.1) and (4.2), integrating over a box, absorbing any constants in $\phi_+(\boldsymbol{k}, r)$ and $\phi_-(\boldsymbol{k}, r)$ as necessary, and using (4.6) and (4.7) we can now write the total energy and charge in the form

$$E = \sum_{\boldsymbol{k}r} \omega_{\boldsymbol{k}r}[\phi_-^\dagger(\boldsymbol{k}, r)\Lambda L_0\phi_-(\boldsymbol{k}, r) - \phi_+^\dagger(\boldsymbol{k}, r)\Lambda L_0\phi_+(\boldsymbol{k}, r)]$$

$$Q = \sum_{\boldsymbol{k}r} [\phi_+^\dagger(\boldsymbol{k}, r)\Lambda L_0\phi_+(\boldsymbol{k}, r) + \phi_-^\dagger(\boldsymbol{k}, r)\Lambda L_0\phi_-(\boldsymbol{k}, r)].$$

The quantization procedure must ensure that E is a sum of positive contributions and Q is of the form $\Sigma(N-M)$ where N counts the particles and M counts the antiparticles. Since both of these quantities transform like the fourth components of four-vectors, we can restrict ourselves to the rest frame for these considerations of definiteness.

In the rest frame (4.4) and (4.5) show that $\phi_+(0, r)$ and $\phi_-(0, r)$ are eigenvectors of L_0 corresponding to eigenvalues $+m_r$ and $-m_r$ respectively. Thus $\phi_+(0, r)$ and $\phi_-(0, r)$ can be expanded in eigenvectors of the s blocks:

$$\phi_+(0, r) = \sum_s a_{0rs}\phi_+(0, r, s), \qquad \phi_-(0, r) = \sum_s \bar{b}_{0rs}\phi_-(0, r, s).$$

$\phi_+(0, r, s)$ and $\phi_-(0, r, s)$ are eigenvectors of the s block corresponding to eigenvalues $+m_r$ and $-m_r$ respectively.

Since Λ does not mix up the s blocks of L_0, we can now write

$$\phi_+^\dagger(0, r)\Lambda L_0\phi_+(0, r) = \sum_s \bar{a}_{0rs}a_{0rs}\phi_+^\dagger(0, r, s)\Lambda_s A_s\phi_+(0, r, s)$$

$$\phi_-^\dagger(0, r)\Lambda L_0\phi_-(0, r) = \sum_s b_{0rs}\bar{b}_{0rs}\phi_-^\dagger(0, r, s)\Lambda_s A_s\phi_-(0, r, s)$$

where Λ_s, A_s are the s blocks of Λ and L_0. The total energy and charge can thus be written in the rest frame:

$$E = \sum_{rs} \omega_{0r}\epsilon_+(0, r, s)\left(b_{0rs}b_{0rs}^+\frac{\epsilon_-(0, r, s)}{\epsilon_+(0, r, s)} - a_{0rs}^+a_{0rs}\right)$$

$$Q = \sum_{rs} \epsilon_+(0, r, s)\left(b_{0rs}b_{0rs}^+\frac{\epsilon_-(0, r, s)}{\epsilon_+(0, r, s)} + a_{0rs}^+a_{0rs}\right)$$

where

$$\epsilon_+(0, r, s) = \phi_+^\dagger(0, r, s)\Lambda_s A_s\phi_+(0, r, s) \tag{4.8}$$

$$\epsilon_-(0, r, s) = \phi_+^\dagger(0, r, s)\Lambda_s A_s\phi_-(0, r, s) \tag{4.9}$$

and the a_{0rs}, b_{0rs} are now interpreted as operators in the Hilbert space of states and $+$ denotes the adjoint in this Hilbert space. So the definiteness properties of the quantized energy and charge depend ultimately on the quantities $\epsilon_+(0, r, s)$ and $\epsilon_-(0, r, s)$. The usual spin-statistics connections will be satisfied if, and only if, in the case of:

(i) *Integral spin fields.* $\epsilon_+(0, r, s)$ have the same sign for all r, s and in each case $\epsilon_-(0, r, s)$ has the opposite sign to $\epsilon_+(0, r, s)$.

(ii) *Half-odd integral spin fields.* $\epsilon_+(0, r, s)$ have the same sign for all r, s and in each case $\epsilon_-(0, r, s)$ has the same sign as $\epsilon_+(0, r, s)$.

These are the final conditions we impose on the s blocks in order to get good quantizable theories.

Now let A be any $n \times n$ s block. Then from the last section, the minimal polynomial of A must have the form (3.2) and so A must satisfy the polynomial equation

$$f(A) = A^p \prod_{i=1}^{k} (A^2 - m_i^2) = 0, \qquad m_i \neq 0 \tag{4.10}$$

for any $p \geqslant q$. The reason for considering the polynomial $f(t)$, instead of the minimal polynomial $m(t)$ is that while it is straightforward to find an s block satisfying some polynomial equation, it is very difficult to ensure that this is the *minimal polynomial*, which may be one of lesser degree. In this section $f(t)$ can be any polynomial which annihilates the s block A, but need not necessarily be the minimal polynomial.

Now consider the polynomial $P_{+r}(A)$ such that

$$(A - m_r)P_{+r}(A) \equiv f(A) = 0. \tag{4.11}$$

Since $P_{+r}(A)$ lacks an irreducible factor, $A - m_r$, of the minimal polynomial, it is non-zero, and by definition its columns are either zero or eigenvectors of A corresponding to the eigenvalue m_r. But by the discussion of § 3 and the form of the minimal and characteristic polynomial of A, there is only one such independent eigenvector and so the rank of $P_{+r}(A)$ is unity, and if ψ is an arbitrary vector in the s subspace, then

$$\psi_{+r} = P_{+r}(A)\psi \tag{4.12}$$

is up to a factor the eigenvector of A corresponding to eigenvalue m_r. Similarly, if $P_{+r}(A)$ is the matrix polynomial such that $f(A) \equiv (A + m)P_{-r}(A)$, then the rank of $P_{-r}(A)$ is unity and

$$\psi_{-r} = P_{-r}(A)\psi \tag{4.13}$$

is up to a factor the eigenvector of A corresponding to eigenvalue $-m_r$. For the s block we therefore have

$$\begin{aligned} \epsilon_+(0, r, s) &= \psi_{+r}^{\dagger} \Lambda A \psi_{+r} = \psi^{\dagger} P_{+r}^{\dagger}(A) \Lambda A P_{+r}(A)\psi \\ &= \psi^{\dagger} \Lambda A P_{+r}(A) P_{+r}(A)\psi \\ &= m_r P_{+r}(m_r)\psi^{\dagger} \Lambda P_{+r}(A)\psi. \end{aligned} \tag{4.14}$$

So, since m_r can be assumed positive:

$$\operatorname{sgn}(\epsilon_+(0, r, s)) = \operatorname{sgn}(P_{+r}(m_r)) \cdot \operatorname{sgn}(\psi^{\dagger} \Lambda P_{+r}(A)\psi). \tag{4.15}$$

Now consider $\operatorname{sgn}(\psi^{\dagger} \Lambda P_{+r}(A)\psi)$. Since Λ is non-singular,

$$\operatorname{rank}(\Lambda P_{+r}(A)) = \operatorname{rank}(P_{+r}(A)) = 1,$$

so $\Lambda P_{+r}(A)$ has exactly one nonzero eigenvalue, which is real, since $\Lambda P_{+r}(A)$ is hermitian. $\psi^{\dagger} \Lambda P_{+r}(A)\psi$ will be positive or negative semi-definite according as this single eigenvalue, given by the trace of $\Lambda P_{+r}(A)$, is positive or negative. Hence

$$\operatorname{sgn}(\psi^{\dagger} \Lambda P_{+r}(A)\psi) = \operatorname{sgn}[\operatorname{Tr}(\Lambda P_{+r}(A))] \tag{4.16}$$

and (4.15) becomes

$$\operatorname{sgn}(\epsilon_+(0, r, s)) = \operatorname{sgn}(P_{+r}(m_r)) \cdot \operatorname{sgn}[\operatorname{Tr}(\Lambda P_{+r}(A))]. \tag{4.17}$$

Similarly we find

$$\operatorname{sgn}(\epsilon_-(0, r, s)) = -\operatorname{sgn}(P_{-r}(m_r)) \cdot \operatorname{sgn}[\operatorname{Tr}(\Lambda P_{-r}(A))]. \tag{4.18}$$

Noting that

$$P_{-r}(-m_r) = (-1)^{p+1}P_{+r}(m_r) \tag{4.19}$$

and

$$\operatorname{sgn}(P_{+r}(m_r)) = \operatorname{sgn}\left(\prod_{i\neq r}(m_r^2 - m_i^2)\right) = (-1)^{k_r} \tag{4.20}$$

where k_r is the number of $m_i > m_r$, we can write (4.18) as

$$\operatorname{sgn}(\epsilon_-(0, r, s)) = (-1)^{p+k_r}\operatorname{sgn}[\operatorname{Tr}(\Lambda P_{-r}(A))]. \tag{4.21}$$

We now examine the two trace expressions, $\operatorname{Tr}(\Lambda P_{+r}(A))$ and $\operatorname{Tr}(\Lambda P_{-r}(A))$. Consider quantities of the type $\operatorname{Tr}(\Lambda A^l)$ where l is some natural number. Now

$$\operatorname{Tr}(\Lambda A^l) = \sum_{i,j=1}^{n}(\Lambda)_{ij}(A^l)_{ji} \tag{4.22}$$

and $(\Lambda)_{ij} = 0$ unless row i and column j correspond to two mutually conjugate representations on the graph of the s block. The only nonzero elements $(A^l)_{ji}$, on the other hand, are those corresponding to all paths in the s block graph of length l from the representation denoted by j to that denoted by i. It follows therefore from (4.22) that the only nonzero terms in $\operatorname{Tr}(\Lambda A^l)$ are those corresponding to paths of length l between mutually conjugate representations in the s block graph. We therefore have:

(i) *For integral spin s blocks*:

$$\operatorname{Tr}(\Lambda A^l) = 0 \qquad \text{if } l \text{ odd.} \tag{4.23}$$

(ii) *For half-odd integral spin s blocks*:

$$\operatorname{Tr}(\Lambda A^l) = 0 \qquad \text{if } l \text{ even.} \tag{4.24}$$

Using these results we now consider separately the case of integral spin and half-odd integral spin.

4.1. Integral spin

If p odd then

$$\operatorname{Tr}(\Lambda P_{+r}(A)) = \operatorname{Tr}(\Lambda P_{-r}(A)) = T_r(p+1) \tag{4.25}$$

if p even then

$$\operatorname{Tr}(\Lambda P_{+r}(A)) = -\operatorname{Tr}(\Lambda P_{-r}(A)) = m_r T_r(p) \tag{4.26}$$

where

$$T_r(x) = \operatorname{Tr}\left(\Lambda A^x \prod_{i\neq r}(A^2 - m_i^2)\right). \tag{4.27}$$

It now follows from (4.17) and (4.21) that for p odd or even, for integral spin theories of our type, we always have

$$\operatorname{sgn}(\epsilon_+(0, r, s)) = -\operatorname{sgn}(\epsilon_-(0, r, s)). \tag{4.28}$$

So this part of the condition for quantization is always satisfied. Finally all the $\epsilon_+(0, r, s)$ must have the same sign, say positive, for all r, s and from the above discussion, this

occurs if and only if, for each s block A having nonzero eigenvalues:

if p odd

$$\operatorname{sgn}(T_r(p+1)) = (-1)^{k_r} \tag{4.29}$$

if p even

$$\operatorname{sgn}(T_r(p)) = (-1)^{k_r} \tag{4.30}$$

where k_r is the number of m_i greater than m_r, and $T_r(x)$ as defined at (4.27). In the case when there is just one pair of nonzero eigenvalues for the s block then k_r should be taken as zero to obtain the appropriate conditions.

4.2. Half-odd integral spin

In this case, by analogous arguments to the above, we find that the conditions

$$\operatorname{sgn}(\epsilon_+(0, r, s)) = \operatorname{sgn}(\epsilon_-(0, r, s))$$

are always satisfied; while all the $\epsilon_+(0, r, s)$ are of the same sign if and only if:

if p odd

$$\operatorname{sgn}(T_r(p)) = (-1)^{k_r}$$

if p even

$$\operatorname{sgn}(T_r(p+1)) = (-1)^{k_r}.$$

Note that these are the same conditions as for integral spin, except that 'p odd' and 'p even' are interchanged.

So finally, to find good theories for integral spin without repeated representations, or states of zero charge or energy, and for which the eigenstate corresponding to a given mass and spin is unique; it is necessary and sufficient to choose the s blocks so as to have a characteristic polynomial of the form (3.3), to satisfy (4.10) and also (4.29) or (4.30). Similarly for half-odd integral spin. In practice the graphs of the s blocks can be useful in explicitly writing down these conditions in particular cases.

5. Discussion

The study of the quantization of the Gel'fand–Yaglom equations is difficult in general, because of the complicated algebra involved. Amar and Dozzio (1972b) have obtained necessary restrictions on the possible chains or graphs for quantizable theories subject to the diagonalizability of the s blocks. In the case of unique mass the restrictions on the graphs become sufficient also. Thus we get a general graph for theories of given maximum spin, such that any quantizable theory must be constructed from a representation $\mathscr{R}$ corresponding to a subgraph of this graph. For a particular type of theory, we have given specific necessary and sufficient conditions such that any given subgraph corresponds to a quantizable theory.

By counting the number of conditions to be satisfied, and the number of available arbitrary parameters $C^{\pi\pi'}$, it seems unlikely that there are quantizable theories of the type we have been discussing, for spin greater than eight. So it seems possible to list all such theories which are quantizable, while if we want theories for spin greater than eight

we must presumably introduce repeated representations. It is possible to restrict the graphs for good theories further than Amar and Dozzio have done. For example, a representation $\mathscr{R}$ in which the representations corresponding to the maximum spin j are not all linked by a connected path on the j block graph, cannot give a quantizable theory of our general type. This follows because the j block characteristic equation would have repeated nonzero roots, which by (3.3) is forbidden.

To decide whether a particular representation $\mathscr{R}$ can support a good theory, it may not be necessary to go through the details of the analysis we have described. A reasonable idea can be obtained by counting the available $C^{\tau\tau'}$, which depends on the number of branches in the graph of $\mathscr{R}$, and comparing this with the number of conditions to be satisfied, which depends on the required form of the equations (3.3), (4.10) and also on the trace conditions. Examples of quantizable theories and further details of the types of graphs for such theories will be given in a further paper.

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References

Amar V and Dozzio U 1972a *Nuovo Cim.* B **9** 53–63
—— 1972b *Nuovo Cim.* A **11** 87–99
Capri A Z and Shamaly A 1971 *Nuovo Cim.* B **2** 236–53
Cox W 1972 *PhD Thesis* University of Aston in Birmingham
Gel'fand I M, Minlos R A and Shapiro Z Ya 1963 *Representations of the Rotation and Lorentz Group and their Applications* (Oxford: Pergamon)
Harary F 1959 *J. Math. Phys.* **38** 104–11
Udgaonkar B M 1952 *Proc. Ind. Acad. Sci.* A **36** 482–92